metal wing. The addition of composite material in region 1 actually reduced the flutter speed. Flutter calculations using a massless patch showed that the reduction in flutter speed was caused almost entirely by the stiffness of the patch, rather than by the mass of the patch.

The fact that the flutter design in Fig. 2a has a build-up in thickness in a region between the quarter and mid chord suggests that a composite patch covering this region of the strength-designed wing might be more beneficial in reducing the mass addition for flutter prevention than any of the regions considered in Fig. 4. The effects of such a patch, denoted as region 5, is shown in Fig. 5 along with the results from Fig. 4 for a composite patch in region 3 and the minimum-mass all-titanium design. The mass of composite reinforcement required in region 5 is half that required in region 3 and less than one-third the mass addition required by the all-titanium design, even though no optimality criteria or search techniques were used to determine the location or size of the region or the orientation of the fibers; thus even lighter designs may be possible.

These results clearly demonstrate that selective reinforcement of wing surfaces, using judiciously placed filamentary composites, promises sizeable mass savings in the design of advanced aircraft structures.

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Equations of Motion for Flexible Cables

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Introduction

THE literature on the equations of motion of flexible cables (used here to include strings, ropes, etc.) has a long history beginning with Lord Rayleigh in his treatise on the "Theory of Sound."[1] Since that time, the solution of many problems involving towing cables, mooring cables, etc. have appeared. In addition, a set of problems concerning a moving "Threadline" have been examined by various authors[2–4]. In this note a set of equations are presented which describe a more general problem of cable motion. The general formulation contain as special cases many of the previous problems considered.

Description of the problem

Given an inextensible cable and a fixed point P, in space, one wishes to deploy the cable from another point Q in space so that the free end of the cable attaches itself to the fixed point. In particular, Q may be moving in space and one wishes to prescribe the motion of point Q so that the cable reaches the fixed point P. The cable is capable of moving in space and is acted on by gravity and fluid dynamic forces. A portion of the cable emanating from Q is shown in Fig. 1. A generic point on the cable is denoted by

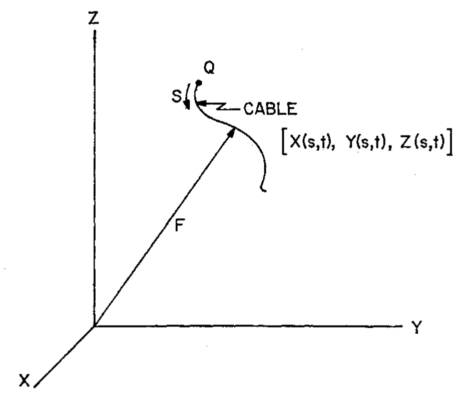

Fig. 1 Cable geometry.

$$x = x(s, t),\ y = y(s, t),\ z = z(s, t) \tag{1}$$

where s is an arc length with respect to Q and t is time. Here x, y, z are coordinates of points on the cables measured in a fixed coordinate system. The equations of motion for the cable is

$$(d/dt)(\rho\bar{q}) = \bar{F} \tag{2}$$

where ρ is the mass per unit length of the cable, $\bar{F}$ is the force per unit length, and $\bar{q}$ is the velocity vector with components V_x, V_y, V_z and

$$\bar{q} = \bar{q}[x(s,t), y(s,t), z(s,t)] \tag{3}$$

The velocity in the x direction for a point on the cable is given by

$$V_x = (dx/dt) = (\partial x/\partial s)(ds/dt) + (\partial x/\partial t) = V(\partial x/\partial s) + (\partial x/dt) \tag{4}$$

where $V = ds/dt$ is the tangential velocity of the cable. In a similar manner, we have the velocity in the y and z directions given, respectively, as

$$V_y = V(\partial y/\partial s) + (\partial y/\partial t),\ V_z = V(\partial z/\partial s) + (\partial z/\partial t) \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (2) and using the total time rate of change operator $(d/dt) = V(\partial/\partial s) + (\partial/\partial t)$ the equations of motion become

$$(\partial/\partial s)[\rho\bar{q}]V + (\partial/\partial t)[\rho\bar{q}] = (\partial/\partial s)[T(\partial\bar{r}/\partial s)] + \bar{f} \tag{6}$$

where $\bar{r} = [x(s,t),\ y(s,t),\ z(s,t)]$ and the forces have been separated into those forces due to the tension T, and the forces due to gravity and fluid dynamics, $\bar{f}$.

To complete the description of motion a continuity equation is needed to account for the change in mass. If a small control volume of cable is considered the rate of mass flow into a region is $\rho V|_s$ and the rate of mass flow out is $\rho V|_{s+\Delta s}$. The rate of change of the total mass within the region is $(\partial\rho/\partial t)\Delta s$. Equating these two and allowing $\Delta s \to 0$ results in

$$(\partial/\partial s)(\rho V) + (\partial\rho/\partial t) = 0$$

or

$$d\rho/dt + \rho\partial V/\partial s = 0 \tag{7}$$

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The description of the problem is completed by a constitutive equation for the density ρ, and the inextensibility condition.

$$\rho = \rho(s,t) \tag{8}$$

$$(\partial\bar{r}/\partial s)\cdot(\partial\bar{r}/\partial s) = 1 \tag{9}$$

The boundary conditions are of the form

$$x(0,t) = X(t),\ y(0,t) = Y(t),\ z(0,t) = Z(t)$$

where $X(t)$, $Y(t)$, and $Z(t)$ describe the motion of point Q. At the fixed point P we have

$$x(L,t) = \tilde{X}[X(t),\ Y(t), Z(t),t]$$

$$y(L,t) = \tilde{Y}[X(t),\ Y(t), Z(t),t]$$

$$z(L,t) = \tilde{Z}[X(t),\ Y(t), Z(t),t]$$

where L is the length of cable to be deployed.

In addition to the above boundary conditions, a condition on the tension T is needed either at $s = 0$ or at $s = L$.

The initial conditions are of the form

$$x(s,0) = x_0(s),\ y(s,0) = y_0(s),\ z(s,0) = z_0(s)$$

$$V_x(s,0) = V_{0x}(s),\ V_y(s,0) = V_{0y}(s),\ V_z(s,0) = V_{0z}(s)$$

where $x_0(s)$, etc. may be found by deploying an initial length of cable $a < L$ and using the steady-state equilibrium configuration.

The solution of the preceding problem falls within the class of problems considered in control theory, i.e., the motion of the moving point is to be found so that the end point of the cable remains at a particular point.

Special cases of cable problems

If we restrict point Q to remain fixed, then a number of well-known problems are described by Eqs. (6–9).

1. One-dimesional vibration of a cable fixed at both ends

Here $V = 0$, $\rho = \text{const}$, $\bar{r} = (x,0,0)$ and Eq. (6) becomes $\rho(\partial^2 x/\partial t^2) = (\partial/\partial s)[T(\partial x/\partial s)] + f$. If the tension T is assumed constant and the effects of gravity and fluid dynamic forces are neglected we have $(\partial^2 x/\partial t^2) = (1/c^2)(\partial^2 x/\partial s^2)$ which is the familiar vibrating string equation.

2. Static equilibrium of a cable configuration

The velocity $V = 0$ and all quantities are independent of time. Equation (6) then yields $(\partial/\partial s)[T(\partial\bar{r}/\partial s)] + \bar{f} = \bar{0}$ which is the equations considered by Alekeev.[5] This last set of equations can be solved by quadratures as shown in Ref. 5.

3. Moving threadline in one dimension

The velocity, $V = V_0$, a constant, the density ρ and the tension T are taken as constants. The force $\bar{f}$ on the cable is neglected and the motion is assumed to be one-dimensional. Equation (6) then becomes

$$\partial^2 x/\partial t^2 + 2V_0\partial^2 x/\partial t\partial s - (c^2 - V_0{}^2)(\partial^2 x/\partial s^2) = 0$$

where $c^2 = T/\rho$. This equation was considered by the authors of Refs. 2–4.

Other equations describing the two- and three-dimensional motion of a cable can also be derived from Eqs. (6) and (7), but are not included in this Note.

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